This is my attempt at a lesson on limits. It's kind of long, but if you look closely, I've helped you out with some of your homework problems, so you might find it worthwhile to read the whole thing. You might even learn how to do these!

Let's start by looking at the equation

$$f(x) = x + 3$$

We want to look at what happens as x gets close to 1. First let's take .5, as that's pretty close to 1. We take $f(.5)$ (which is just the same as taking $x + 3$ and plugging in .5 for x), and we get $f(.5) = .5+3 = 3.5$. Now let's take $x = .9$, which is even closer to 1. $f(.9) = .9 + 3 = 3.9$. Then we can take even closer and closer values to 1 like in the following chart:

x		$f(x)$
.5	$\rightarrow$	$.5 + 3 = 3.5$
.9	$\rightarrow$	$.9 + 3 = 3.9$
.99	$\rightarrow$	$.99 + 3 = 3.99$
.999	$\rightarrow$	$.999 + 3 = 3.999$

You're probably starting to notice a pattern here. As we take values of x closer and closer to 1, our values for $f(x)$ (or you can think of it as y) are getting closer and closer to 4. Then we say that "the limit of $f(x)$ as x approaches 1 is 4," or you can write it like this:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x + 3 = 4$$

That's all a limit is.

Now let's do another example. We will find the limit of $f(x) = x^2 - 2$ as x approaches 4, or

$$\lim_{x \to 4} x^2 - 2$$

So let's pick some values close to 4 and make a chart like we did in the previous example.

x		$f(x)$
3.5	$\rightarrow$	$(3.5)^2 - 2 = 10.25$
3.9	$\rightarrow$	$(3.9)^2 - 2 = 13.21$
3.99	$\rightarrow$	$(3.99)^2 - 2 = 13.9201$
3.999	$\rightarrow$	$(3.999)^2 - 2 = 13.992001$

We can also pick values just above 4 as in this other chart:

x		$f(x)$
4.5	$\rightarrow$	$(4.5)^2 - 2 = 18.25$
4.1	$\rightarrow$	$(4.1)^2 - 2 = 14.81$
4.01	$\rightarrow$	$(4.01)^2 - 2 = 14.0801$
4.001	$\rightarrow$	$(4.001)^2 - 2 = 14.008001$

As you can see, the $f(x)$ values are getting pretty close to 14. Then

$$\lim_{x\to 4} x^2 - 2 = 14$$

Let's look back at the first example where $f(x) = x + 3$. We're taking x values that get closer and closer to 1, so let's try taking $x = 1$ and see what happens (because there isn't a number closer to 1 than 1 itself). $f(1) = 1+3 = 4$. It just so happens that the value of our limit is the value of the function in this case. Now let's look at the second example where $f(x) = x^2 - 2$. We're taking values really close to 4, so let's try plugging in $x = 4$. Then $f(4) = 4^2 - 2 = 14$, which is also what we calculated for the limit. This is not a coincidence. The closer x gets to 4, the closer your y value gets to 14, so it makes sense that when $x = 4$, $y = 14$. Therefore, a good way to solve limits is to just plug the number in.

Now let's look at the example

$$f(x) = \frac{x^2 - 4}{x + 2}$$

and let's take the limit as x approaches 3. Well, as we saw above, let's try plugging in $x = 3$.

$$f(3) = \frac{3^2 - 4}{3 + 2} = \frac{5}{5} = 1$$

Then just like in the previous examples,

$$\lim_{x\to 3} \frac{x^2 - 4}{x + 2} = 1$$

Now let's use this same example but this time calculate

$$\lim_{x\to -2} \frac{x^2 - 4}{x + 2}$$

When we plug in $x = -2$, we see that we get $f(-2) = \frac{0}{0}$. This is what is called *indeterminate form*. It's called that because we don't know what $\frac{0}{0}$ actually is. However, that doesn't mean we should give up on the problem, because there's still something we can do.

Let's take a look at why we're getting $\frac{0}{0}$. This happened because we plugged in $x = -2$. Now we know that $x = -2$ exactly when $x + 2 = 0$, so that must mean that both the numerator and the denominator of the fraction have an $x+2$ in there somewhere. So let's factor the fraction to see if we can find it.

$$f(x) = \frac{x^2-4}{x+2} = \frac{(x+2)(x-2)}{x+2}$$

So we can see that our suspicions were correct, and there is indeed an $x+2$ in both the numerator and the denominator. Now it would make sense just to cancel them. However, there's a problem with that. Canceling the $x+2$ from the top and bottom is essentially the same as dividing both the top and bottom by $x+2$. We know from rules of division that we can't divide by 0, so we can only cancel if we're absolutely sure that $x+2 \neq 0$. That's where our limits can help. If we take

$$\lim_{x\to -2} \frac{(x+2)(x-2)}{x+2}$$

then what we're doing is taking x values really really close to -2, but we're not going to hit -2. That means that $x \neq -2$, or $x+2 \neq 0$, which is exactly what we need in order to cancel. Mind you, $x+2$ is going to get really close to 0, but it's not going to equal 0.

So let's take the limit:

$$\lim_{x\to -2} \frac{(x+2)(x-2)}{x+2} = \lim_{x\to -2} x-2$$

Now we're in good shape, as we know exactly what to do in this situation, and that is plug in $x=-2$. Then we get

$$\lim_{x\to -2} x-2 = (-2)-2 = -4$$

So -4 is our answer. Now let's make a table of values close to $x=-2$ to make sure we did this problem correctly.

x		$f(x)$
-1.5	$\to$	$\frac{(-1.5)^2-4}{-1.5+2} = \frac{-1.75}{0.5} = -3.5$
-1.9	$\to$	$\frac{(-1.9)^2-4}{-1.9+2} = \frac{-0.39}{0.1} = -3.9$
-1.99	$\to$	$\frac{(-1.99)^2-4}{-1.99+2} = \frac{-0.0399}{0.01} = -3.99$
-2.01	$\to$	$\frac{(-2.01)^2-4}{-2.01+2} = \frac{.0401}{-0.01} = -4.01$
-2.1	$\to$	$\frac{(-2.1)^2-4}{-2.1+2} = \frac{0.41}{-0.1} = -4.1$
-2.5	$\to$	$\frac{(-2.5)^2-4}{-2.5+2} = \frac{2.25}{0.5} = -4.5$

Then as x values get closer and closer to -2, we see that our y values get closer and closer to 4, so we know we did something right.

Let's try another example. Find:

$$\lim_{x\to 2} \frac{2x+1}{x+2}$$

Well, as we've done previously, our first step is to plug in $x = 2$ and see what happens. When we do that, we get

$$f(2) = \frac{5}{4}$$

So in fact, we're done. We just found that

$$\lim_{x\to 2} \frac{2x+1}{x+2} = \frac{5}{4}$$

Let's try another. Find:

$$\lim_{x\to -2} \frac{x^2-x-6}{x^2+x-2}$$

Well, again, we plug in $x = -2$ to see what we get.

$$f(-2) = \frac{4+2-6}{4-2-2} = \frac{0}{0}$$

So we have an indeterminate form, which means we need to do a little work. Remember that $x = -2$ exactly when $x + 2 = 0$, so if there's a 0 on the top and bottom when $x = -2$, that means we have an $x + 2$ on the top and bottom. So let's factor.

$$\frac{x^2-x-6}{x^2+x-2} = \frac{(x-3)(x+2)}{(x+2)(x-1)}$$

Now, remember we can't cancel the $x + 2$ unless we can be sure that $x + 2 \neq 0$. But if we're taking a limit as $x \to -2$, we're taking values of x really close to -2, but not equal to -2. Then $x \neq -2$, so $x+2 \neq 0$. That means we can cancel as long as we remember that we're taking a limit.

$$\lim_{x\to -2} \frac{x^2-x-6}{x^2+x-2} = \lim_{x\to -2} \frac{(x-3)(x+2)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{x-3}{x-1}$$

Now we've canceled the terms that were giving us $\frac{0}{0}$, so when we plug in -2 again, we're not going to get $\frac{0}{0}$ anymore. So let's do that:

$$\lim_{x\to -2} \frac{x-3}{x-1} = \frac{(-2)-3}{(-2)-1} = \frac{-5}{-3} = \frac{5}{3}$$

And that's our answer. If you're not convinced, you should make a table of values of x near -2 just to double check.

Now let's do a slightly different problem. Let's find

$$\lim_{x\to 0} \frac{1}{x}$$

When we plug in 0, we get $\frac{1}{0}$, which means we have to be careful. 0 is the point where numbers change from positive to negative and vice versa, so when you have a number over 0, you have to worry about which side of 0 you're on. That probably doesn't make any sense to you, so let's make a chart of values near 0:

x		$f(x)$
.1	$\rightarrow$	10
.01	$\rightarrow$	100
.001	$\rightarrow$	1000

So when we choose x values just above 0, as the number gets closer to 0, its y value gets larger and larger, meaning we're approaching ∞. However, let's look at numbers just below 0:

x		$f(x)$
$-.1$	$\rightarrow$	-10
$-.01$	$\rightarrow$	-100
$-.001$	$\rightarrow$	-1000

As you can see, on this side, we're approaching $-\infty$. So we run into the problem, if we want to find

$$\lim_{x \to 0} \frac{1}{x}$$

do we say that this is ∞ or $-\infty$? How do we choose? Well, mathematics is nice in that it doesn't let you choose. This is when you say that *the limit does not exist*. Let's do another example. Find:

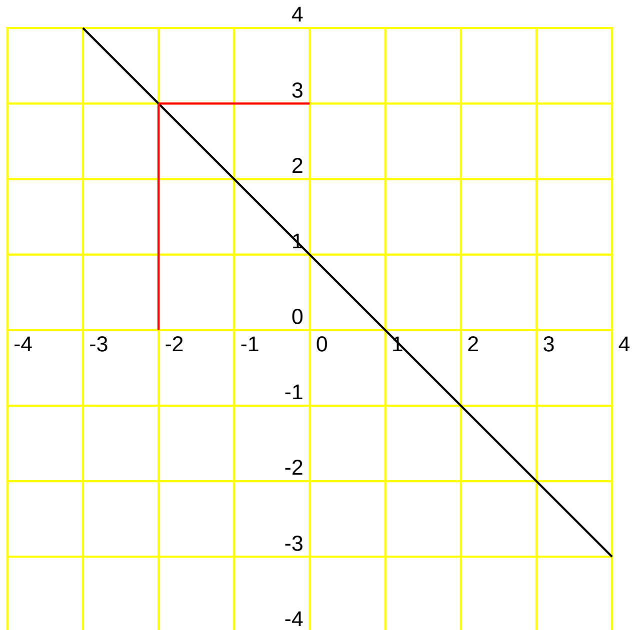
$$\lim_{x \to 1} \frac{x}{x^2 - x}$$

As we know, the way you start all limit problems is to plug in $x = 1$ and see what you get. So we get

$$\frac{1}{1^2 - 1} = \frac{1}{0}$$

Then we're back in the same situation as before. You can make a chart for yourself, but if we plug in numbers just above 1, we're going to get that $\frac{x}{x^2-x}$ is approaching ∞. However, if we plug in numbers just below 1, the denominator is going to be negative, so we are approaching $-\infty$. Then once again, we're left with the problem of which one to choose, so we say that the limit does not exist.

Now let's look at what limits mean in terms of a graph. Let's look at this graph of $y = f(x)$:



Sorry about the terrible colors. Anyway, let's find

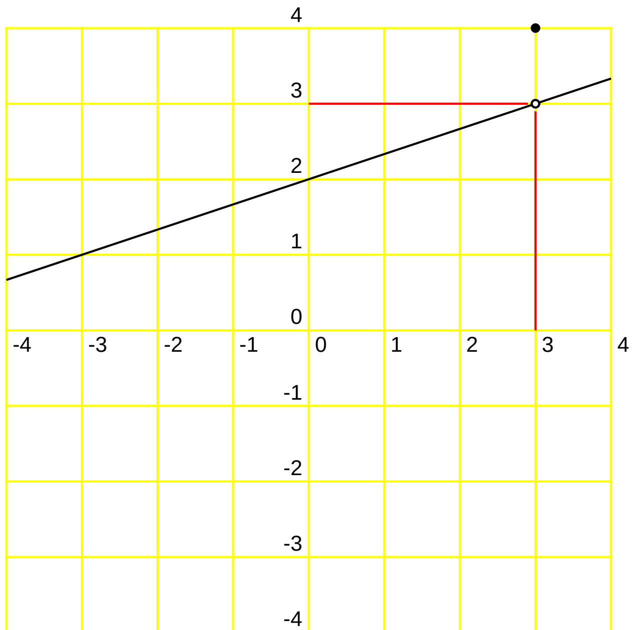
$$\lim_{x \to -2} f(x)$$

Well, let's handle this problem exactly the same as we've handled the previous ones. Let's look at x values close to $x = -2$. Let's start with x values greater than -2. At $x = -1$, $f(-1) = 2$. Then as we move closer to -2, let's say $x = -1.5$, then it looks like $f(x)$ is about 2.5. Moving even closer, we see that $f(x)$ (or your y value) is getting closer and closer to 3. Now let's look at x values less than -2. Let's take $x = -2.5$. Then it looks like the y value is around 3.5. Moving x closer and closer to -2, we see that $f(x)$ is getting closer and closer to 3. Then we can say that

$$\lim_{x \to -2} f(x) = 3$$

This is exactly the same way as we did the problems earlier, but instead of having a chart of points, we have a picture instead, and a picture is just a whole bunch of points.

NOTE: We did not care what the y value is at $x = -2$. That's not what a limit asks. A limit asks you to look at points near $x = -2$ and predict a pattern. A limit does *not* ask you what the value of y is at a certain point. That's something you learned in elementary school, and if limits were the same thing as something you did in elementary school, you would have done limits back then. The value of y here could have been 3 or $247,000$. It doesn't matter. The point is to predict the pattern using values of x near -2. It just so happens that the value of f at $x = -2$ is 3, but again that doesn't matter. Let's do another example to prove it:



Let's find

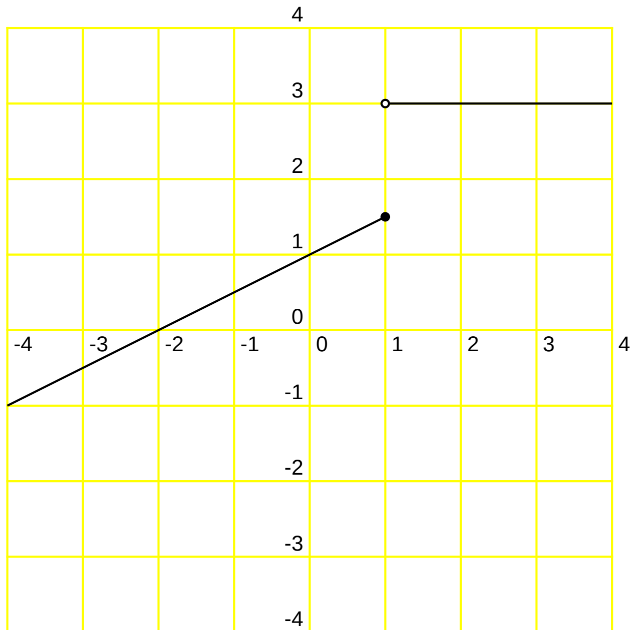
$$\lim_{x \to 3} f(x)$$

Again, let's take x values close to $x = 3$. If we use x values just below 3, for example, $x = 2$, we get that $f(x)$ is about $2\frac{2}{3}$. Then as we move closer and closer to $x = 3$, y gets closer and closer to 3. Now let's look at x values bigger than 3, say $x = 4$. Then it looks like y is about $3\frac{1}{3}$. Then as x gets closer and closer to 3, y gets closer and closer to 3. So we say that

$$\lim_{x \to 3} f(x) = 3$$

Now if you look at the graph, you'll notice a point at $(3, 4)$, meaning $f(3) = 4$. But we just calculated that $\lim_{x \to 3} = 3$. Did we do something wrong? The answer is no. Remember, as I said before, we DO NOT care that at $x = 3$, $y = 4$. That's not what the limit means. The limit is exactly what we've been calculating all along, and it may or may not have anything to do with the actual y value at that x coordinate.

Now let's do yet another example:



Let's find

$$\lim_{x \to 1} f(x)$$

Let's start with x values less than 1. If $x = 0$, then $y = 1$, and following the line, as we get closer and closer to $x = 1$, we see that y gets closer and closer to somewhere around 1.5. Now let's look at x values bigger than 1. For example, let's take $x = 2$. Then $y = 3$. As we take values closer and closer to $x = 2$, we travel along the flat line, so we're pretty much going to stay at $y = 3$. Then what's the limit? Well, we have a problem because we get two different answers. Coming from the left, we get a y value of 1.5, but from the right, we get a y value of 3. Well, this should remind you a little of an example we did previously (look back at $y = \frac{1}{x}$). So if we have two values that could be the limit, and we don't know how to choose between them, mathematics doesn't let us choose, and we just say that the limit does not exist.

Well, that's about it for this lesson. Thanks for reading it, and I hope it helped.