Lognormal Distributions of Aerosols

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Lognormal distributions are commonly used in a variety of statistical applications. In the field of atmospheric science, one common use is to describe size distributions of stratospheric aerosols. However, the terminology used can often be ambiguous and confusing, often depending upon notation. I am not attempting to standardize the definitions and formulas used for lognormal distributions. I'm merely trying to illustrate the math and various equivalences so that when you see a formulation of a lognormal distribution, you'll know what they're doing.

1 Definitions

A **lognormal distribution** is a probability distribution for which the log is normally distributed. So let's start with the definition of a normal distribution (equation 1) of aerosol radii (r) with mean μ and standard deviation σ:

$$n(r) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(r-\mu)^2}{2\sigma^2}\right] \tag{1}$$

Then a lognormal distribution (equation 2) of aerosol radii ($r > 0$) is given by:

$$n(r) = \frac{1}{r\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\ln(r)-\mu)^2}{2\sigma^2}\right] \tag{2}$$

where μ and σ are the mean and standard deviation of the associated normal distribution. Note that μ and σ are NOT the mean and standard deviation of the lognormal distribution. If we say that $\overline{r}$ and s are the mean and standard deviation (also called distribution width) of the lognormal distribution, then we have the following formulas (equations 3 and 4) that relate these quantities to μ and σ:

$$\mu = \ln\left(\frac{\overline{r}^2}{\sqrt{\overline{r}^2 + s^2}}\right) \tag{3}$$

$$\sigma^2 = \ln\left[1 + \frac{s^2}{\overline{r}^2}\right] \tag{4}$$

Converting the other direction, we have formulas 5 and 6:

$$\overline{r} = e^{\mu+\sigma^2/2} \tag{5}$$

$$s^2 = \left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2} \tag{6}$$

Two more useful quantities are the median (r_{med}) and mode ($\dot{r}$) radius of the lognormal distribution. In a normal distribution, all of these coincide. However, in a lognormal distribution, $\dot{r} < r_{\text{med}} < \overline{r}$. More specifically, $r_{\text{med}} = e^{\mu}$ and $\dot{r} = e^{\mu-\sigma^2}$, where again, μ and σ are the mean and standard deviation of the associated normal distribution.

2 Geometric Moments

Instead of using the arithmetic mean and standard deviations of the lognormal distribution, it is often more convenient to use the geometric mean and standard deviation. These are given by the simple formulas $\overline{r}_g = e^{\mu}$ and $\sigma_g = e^{\sigma}$. Note that $\overline{r}_g = r_{\text{med}}$.

Relating the arithmetic and geometric means,

$$\frac{\overline{r}_g}{\overline{r}} = \frac{e^{\mu}}{e^{\mu+\sigma^2/2}} = e^{-\sigma^2/2}$$

$$\overline{r}_g = \overline{r} e^{-\sigma^2/2} \tag{7}$$

Substituting these definitions into equation 2, we get

$$n(r) = \frac{1}{r\sqrt{2\pi\{\ln(\sigma_g)\}^2}} \exp\left[-\frac{\{\ln(r) - \ln(\overline{r}_g)\}^2}{2\{\ln(\sigma_g)\}^2}\right] \tag{8}$$

which is a more common formulation used in aerosol literature.

3 Mode radius

Formulating a lognormal distribution in terms of the mode radius, while perhaps more intuitive to think about (the mode radius is the point at which the distribution peaks), is mathematically more difficult. There are many manipulations that can be done, but we outline one for which $\dot{r}$ and s are known.

By definition, $\dot{r} = e^{\mu-\sigma^2} = \overline{r} e^{-(3/2)\sigma^2}$. Rearranging,

$$\frac{\overline{r}}{\dot{r}} = e^{(3/2)\sigma^2}$$

Using equation 4 and some slight manipulation, we get the relation

$$\frac{\overline{r}}{\dot{r}} = \left(1 + \frac{s^2}{\overline{r}^2}\right)^{3/2} \tag{9}$$

This formulation is difficult to solve explicitly, but in most cases involving aerosol distributions, a numerical solution to this relation is perfectly acceptable. I recommend solving it numerically (or even graphically, if you prefer). Now $\overline{r}$, $\dot{r}$, and s are known, so by equations 3 and 4, all other relevant quantities are known, and the distribution can be calculated.

4 Aerosol effective radius

A useful quantity in aerosol science is the area-weighted radius, also called the **aerosol effective radius** (denoted r_{eff}). It is defined by the ratio of the third and second moments of the distribution, or

$$r_{\text{eff}} = \frac{\int r^3 n(r)\, dr}{\int r^2 n(r)\, dr}$$

A more succinct formulation is given in terms of the geometric mean and standard deviation:

$$r_{\text{eff}} = \overline{r}_g \exp\left[\frac{5}{2}\left\{\ln(\sigma_g)\right\}^2\right] \tag{10}$$

5 Multi-modal distributions

Until this point, the discussion has been for distributions with a single mode. However, aerosol distributions can often be better described by a bimodal (or in rare cases a trimodal) distribution.

$$n(r) = \sum_{i=1}^{m} \frac{N_i}{r\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{(\ln(r) - \mu_i)^2}{2\sigma_i^2}\right] \tag{11}$$

where i denotes the ith mode of the distribution (out of m total modes), μ_i and σ_i are the mean and standard deviation of the associated normal distribution for each mode, and N_i is the appropriate weight for each of the modes. All of the calculations in the previous sections can be done for each mode.

The definition of the aerosol effective radius needs to be generalized, as the definitions of the second and third moments become more complicated:

$$r_{\text{eff}} = \frac{\sum_{i=1}^{m} N_i \overline{r}_{g_i}^3 \exp\left[\frac{9}{2}\left\{\sigma_{g_i}\right\}^2\right]}{\sum_{i=1}^{m} N_i \overline{r}_{g_i}^2 \exp\left[2\left\{\sigma_{g_i}\right\}^2\right]} \tag{12}$$

It can be seen rather easily that for a unimodal distribution ($m = 1$), equation 12 reduces to equation 10.